Statistical evolution of 3-geometries
– A game dynamics –

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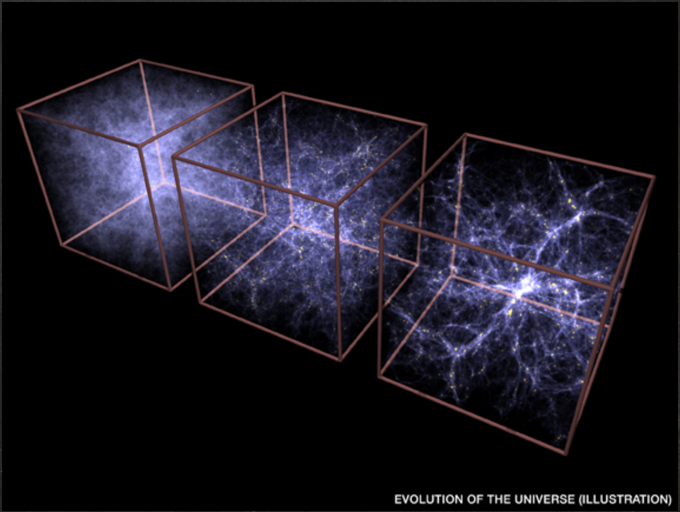
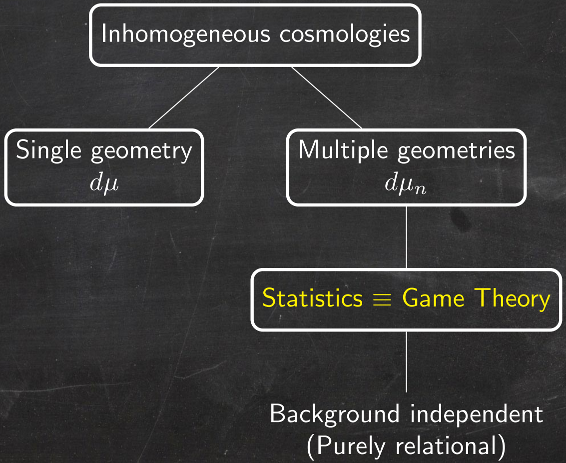
What?



$$R_{\text{voids}} \neq R_{\text{filaments}} \neq R_{\text{clumps}}$$

http://chandra.harvard.edu/chronicle/0408/darkenergy/

What?


Inhomogeneous cosmologies
Single geometry
$d\mu$
Multiple geometries
$d\mu_n$
Statistics $\equiv$ Game Theory
Background independent
(Purely relational)

An analogy

Ecology

- Species
- Population proportions
- Population evolution
- Replicator equation

General relativity

- Spatial curv. types
- Volume proportions
- Volume evolution
- Evol. volume fractions

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Ecology

- n: species index
- N_n: nbr. of individuals
- Total nbr.

$$N_T = \sum_n N_n$$

- Population proportions

$$p_n = \frac{N_n}{N_T}$$

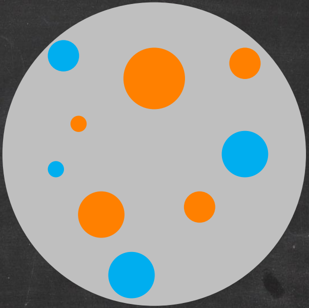
- Population evolution

$$\frac{dN_n}{dt} = f_n(.)N_n$$

- Replicator equation

$$\frac{dp_n}{dt} = \left(f_n - \bar{f}\right) p_n, \quad \bar{f} = \sum_m f_m p_m$$

Cosmology



$$D_n = \bigcup_i D_{n_i},$$

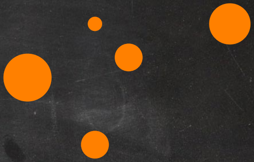
$$D_T = \bigcup_n D_n.$$

$$ds_n^2 = -\alpha_n^2 dt_n^2 + h_{\mu\nu}^n \left(dx_n^\mu + \beta_n^\mu dt_n\right)\left(dx_n^\nu + \beta_n^\nu dt_n\right) .$$

How?

$$\frac{\partial \ln\sqrt{h_n}}{\partial t_n} = -\alpha_n K_n + D^n_\mu \beta^\mu_n = f_n \qquad \text{(evol. 3 – metric)}$$

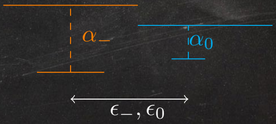
$$\sum_i \int_{D_{n_i}} \frac{\partial\sqrt{h_n}}{\partial t_n} d^3x_n = \sum_i \int_{D_{n_i}} f_n \sqrt{h_n} d^3x_n.$$



$$\frac{\partial V_n}{\partial t_n} = \langle f_n \rangle V_n, \quad \langle f_n \rangle = \frac{\sum_i \int_{D_{n_i}} f_n \sqrt{h_n} d^3x_n}{V_n}.$$

$$\left(d\tau_n = \underbrace{\alpha_n}_{\text{int.}} dt_n = \underbrace{\epsilon_n}_{\text{ext.}} dt \right), \quad \frac{\alpha_n dt_n}{\alpha_m dt_m} = \frac{\epsilon_n \cancel{dt}}{\epsilon_m \cancel{dt}}$$

$$\frac{\partial V_n}{\partial t} = \langle F_n \rangle V_n, \quad \langle F_n \rangle = \frac{\epsilon_n}{\alpha_n} \langle f_n \rangle \rightarrow \text{Fitness}$$



How?

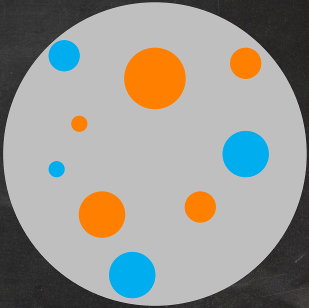
Volume evolution

$$\frac{\partial V_n}{\partial t} = \langle F_n \rangle V_n, \qquad p_n = \frac{V_n}{\sum_m V_m}$$

$$\langle\langle F \rangle\rangle = \sum_m \langle F_m \rangle p_m$$

Replicator equation

$$\frac{\partial p_n}{\partial t} = \left(\langle F_n \rangle - \langle\langle F \rangle\rangle \right) p_n$$



Game theory and Replicator eq.

- Payoff matrix - Fitness

$$F_n = \sum_r \pi_{nr} p_r .$$

- Rock-Paper-Scissors

$\hat{\pi} =$

	Rock	Paper	Scissors
Rock	$(0, 0)$	$(-1, 1)$	$(1, -1)$
Paper	$(1, -1)$	$(0, 0)$	$(-1, 1)$
Scissors	$(-1, 1)$	$(1, -1)$	$(0, 0)$

$$F_{Rock} = 0 \times \frac{1}{3} + (-1) \times \frac{1}{3} + (+1) \times \frac{1}{3}$$

Nash equilibrium and ESS

- Nash equil. $\rightarrow$ Max payoff $\forall\ i, j$
- Evol. stable strategy $\rightarrow$ Fixed points
- GR

$$\frac{\partial p_n}{\partial t} = \left(\langle F_n \rangle - \langle\!\langle F \rangle\!\rangle\right) p_n = 0$$

$$\langle F_n \rangle = \langle\!\langle F \rangle\!\rangle \ \rightarrow \ F_- = F_0 = F_+$$

"Geometric statistical homogeneity"

Example: Timescape

$$\text{n : walls, voids.}$$

$$\alpha_n = 1,\ \beta_n = 0,\ t \to \bar{t},\ \epsilon_n = \{1/\gamma_w, 1/\gamma_v\}$$

For voids,

$$\langle F_v \rangle = H_v(\bar{t}) = \frac{1}{a_v}\frac{da_v}{\gamma_v d\tau_v}$$

$$\langle\!\langle F \rangle\!\rangle = \bar{H} = p_v H_v + p_w H_w$$

$$\frac{\partial p_v}{\partial \bar{t}} = \frac{dp_v}{d\bar{t}} = 3 p_v p_w \left(H_v - H_w\right)$$

$$\frac{dp_v}{d\bar{t}} = 0 \to H_v = H_w \to \text{Uniform Hubble flow!}$$

$$\text{Non} - \text{autonomous system!}$$

Relative information entropy

Kullback-Leibler divergence

$$D_{KL}\left(p^*||p\right) = \sum_{n=1} p_n(t^*) \ln\left(\frac{p_n(t^*)}{p_n(t)}\right).$$

$$\frac{dD_{KL}\left(p^*||p\right)}{dt} < 0$$

Lyapunov function

$$D_{KL}\left(p^*||p\right)\Big|_{t^*} = 0$$

(Relative "relative" information)

Information entropy

$$D_{KL}\left(p^*||p\right) = -\underbrace{\left(-\sum_n p_n(t)\ln p_n(t)\right)}_{\text{Self/Gibbs/Shannon entropy}}\Bigg|_{t=t^*} + \underbrace{\left(-\sum_n p_n(t^*)\ln p_n(t)\right)}_{\text{Cross entropy}}$$

$$S = -\sum_n p_n(t)\ln p_n(t)$$

$$\left.\frac{\partial S}{\partial t}\right|_{t^*} = 0 \;\; \rightarrow \;\; S_{\max}$$

Fixed points $\rightarrow$ Geometric statis. homogeneity

The rest?

- Matrix notation → Grassmann manifold

$$G_m(H) := \{\hat{P} \in Hom(H) : \hat{P} = \hat{P}^2, \hat{P} = \hat{P}^\dagger, Tr\hat{P} = m \in \mathbb{Z}_+\},$$

$$Tr\hat{P} = 1 \rightarrow \text{ Pure states}, \quad \hat{\mathrm{P}} = |\Psi\rangle \langle\Psi|$$

$$|\Psi\rangle = \sqrt{p_-}|-\rangle + \sqrt{p_0}|0\rangle + \sqrt{p_+}|+\rangle,$$

- Payoff matrix ($\hat{\pi}$)→ Fitness matrix ($\hat{F}$), $F_{nn=} \sum \pi_{nr} p_r$
- Lax-type replicator equation

$$\frac{\partial \hat{P}}{\partial t} = \left[\hat{\Lambda}, \hat{P}\right], \quad \hat{\Lambda} = \left[\hat{F}, \hat{P}\right]$$

$$i\frac{\partial \hat{P}}{\partial t} = \left[\hat{H}, \hat{P}\right], \quad \hat{H} = i\Lambda$$

- SU(n) structure → Evol. on Bloch sphere.

Summary

- Inhomogeneous cosmo. $\rightarrow$ multiple geometries.
- Statistical evol. $\rightarrow$ replicator eq. $\rightarrow$ game theory.
- Fixed points $\rightarrow$ Nash equil. $\rightarrow$ an attractor
- Max. self information entropy $\rightarrow$ geometric statis. hom.
- The rest:
  Non-linear, non-autonomous system, with constraints.
  Light propagation? Optical matrix: $\hat{O} \rightarrow \left\langle\!\left\langle \hat{O} \right\rangle\!\right\rangle = Tr\left(\hat{O}\hat{P}\right)$
  Boundary conditions?