A non-perturbative approach to inhomogeneous cosmology

Harald Skarke

Theoretische Physik
TU Wien

July 2017 / Torun

based on
PRD89 (2014) 043506 (arXiv:1310.1028),
PRD90 (2014) 063523 (arXiv:1407.6602),
EPJC77 (2017) no.3, 177 (arXiv:1508.01510).

Motivation and basics

- Motivation:
  - Do we need $\Lambda \neq 0$?
  - Why is that so hard to find out?
- What can one study?
  - Toy models vs realistic universe
    (here: only the latter)
  - Volume evolution or light propagation
- Basic idea:
  - Consider a large domain $\mathcal{D}$ in an irrotational dust universe,
  - divide $\mathcal{D}$ into "infinitesimal" regions — actually
    - small in cosmic terms,
    - large enough for the irrotational dust approximation,
  - follow the evolution of each such region,
  - add the contributions.

Related work

- Many proposals for alternatives to Λ (see e.g. Ellis et al for a review).
- Work on volume evolution:
  - authors such as
    - Kolb, Matarrese, (Notari,) Riotto;
    - Räsänen;
    - Li, (Seikel,) Schwarz;

    (see Buchert for a review),
  - acceleration as a real effect in an irrotational dust universe,
  - Buchert's formalism — here my approach is different!
- Light propagation: redshift-distance relation up to second order perturbation theory worked out by
  - Ben-Dayan, Gasperini, Marozzi, Nugier, Veneziano;
  - Umeh, Clarkson, Maartens.

Outline

Irrotational dust: general setup

- Spacetime manifold ${}^{(4)}\mathcal{M} = \mathbb{R}_+ \times \mathcal{M}$
- Energy-momentum tensor ${}^{(4)}T = \mathrm{diag}(\rho, 0, 0, 0)$
- Metric $ds^2 = -dt^2 + g_{ij}(t, x)dx^i dx^j$ in synchronous gauge, any geometric quantity (e.g. R^i_j) refers to the 3d-metric g!
- Expansion tensor $\theta^i_j = \frac{1}{2} g^{ik} \dot{g}_{kj}$
- Scalar expansion rate $\theta = \theta^i_i = \frac{\dot{\sqrt{g}}}{\sqrt{g}}$
- Shear $\sigma^i_j = \theta^i_j - \frac{\theta}{3}\delta^i_j$

Local scale factor

$a_{\rm local}(t,x) = \left(\frac{\hat\rho}{\rho(t,x)}\right)^{1/3}$ ... the local scale factor
(with $\hat\rho$ ... a fixed quantity of dimension mass, e.g. $M_\odot$);

$a(t,x) = a_{\rm local}(t,x)$ is just the side length of a cube of mass $\hat\rho$ consisting of material of density ρ.

Mass-weighted average

Energy-momentum conservation $\Rightarrow \frac{d}{dt}\left(\rho(x,t)\sqrt{g(x,t)}\right) = 0$

$\Rightarrow$ the mass content $m_{\mathcal{D}} = \int_{\mathcal{D}} \rho(x,t)\sqrt{g(x,t)}\, d^3x$ of any domain $\mathcal{D} \subset \mathcal{M}$ is time independent, $\dot{m}_{\mathcal{D}} = 0$

$\Rightarrow \langle X \rangle^{\cdot}_{\mathrm{mw}} = \langle \dot{X} \rangle_{\mathrm{mw}}$ for the mass–weighted average of a scalar X,

$$\langle X \rangle_{\mathrm{mw}}(t) = \frac{1}{m_{\mathcal{D}}} \int_{\mathcal{D}} X(x,t)\rho(x,t)\sqrt{g(x,t)}\, d^3x,$$

N.B. $\langle X \rangle^{\cdot} = \langle \dot{X} \rangle$ would not hold for a volume average $\langle X \rangle_{\mathrm{vol}}$ which would therefore require Buchert's formalism!

But $V_{\mathcal{D}} = \int_{\mathcal{D}} \sqrt{g(x,t)}\, d^3x = m_{\mathcal{D}} \langle \rho^{-1} \rangle_{\mathrm{mw}} = \frac{m_{\mathcal{D}}}{\hat{\rho}} \langle a^3 \rangle_{\mathrm{mw}}$,
$\langle X \rangle_{\mathrm{vol}} = \langle X a^3 \rangle_{\mathrm{mw}} / \langle a^3 \rangle_{\mathrm{mw}}$ with $a = a_{\mathrm{local}}$.

Rescaled quantities and their evolution

- Rescaled quantities

$$\hat{\rho} = a^3 \rho, \quad \hat{\sigma}^i_j = a^3 \sigma^i_j, \quad \hat{R} = a^2 R, \quad \hat{r}^i_j = a^2 r^i_j$$

(with r^i_j ... traceless part of R^i_j)

- obey the evolution equations

$$\dot{\hat{\rho}} = 0, \qquad \dot{\hat{\sigma}}^i_j = -a \hat{r}^i_j, \qquad \dot{\hat{R}} = -2a^{-3} \hat{\sigma}^i_j \hat{r}^j_i,$$

$$\dot{\hat{r}}^i_j = a^{-3} \left(-\frac{5}{4} \hat{\sigma}^i_k \hat{r}^k_j + \frac{3}{4} \hat{\sigma}^k_j \hat{r}^i_k + \frac{1}{6} \delta^i_j \hat{\sigma}^k_l \hat{r}^l_k \right) + a^2 Y^{ki}{}_{j|k},$$

$$\text{with } Y^k{}_{ij} = \frac{3}{4} (\sigma^k_{i|j} + \sigma^k_{j|i}) - \frac{1}{2} g_{ij} \sigma^k_{m|}{}^m - \sigma_{ij|}{}^k .$$

Local Friedmann equation

The evolution equation for the local scale factor (the 0-0 Einstein equation) can be written as

$$\dot{a}^2 = \frac{1}{3}\hat{\sigma}_{\rm in}^2 a^{-4} + \frac{8}{3}\pi G_N \hat{\rho}\, a^{-1} - \frac{1}{6}\hat{R}_{\rm in} + \frac{1}{3}\Lambda\, a^2 - \frac{4}{9}\int_{t_{\rm in}}^{t} \theta(\tilde{t}) a^{-4}(\tilde{t})\, \hat{\sigma}^2(\tilde{t})\, d\tilde{t}.$$

with $t_{\rm in}$ ... some initial time (typically $t_{\rm in} = 0$).

Linear perturbation theory: metric

- Background: Einstein – de Sitter, i.e. flat matter–only, $a_{\mathrm{EdS}} = \text{const} \times t^{\frac{2}{3}}$.
- The relevant contributions all come from a single time-independent function $C(x)$:
$$g_{ij}^{(\mathrm{LPT})}(t, x) = a_{\mathrm{EdS}}^2(t) \left(\delta_{ij} + \frac{10}{9} \frac{a_{\mathrm{EdS}}^2}{t^{\frac{4}{3}}} C(x) \delta_{ij} + t^{2/3} \frac{\partial^2 C}{\partial x^i \partial x^j} \right).$$
- Corresponds to Newtonian (longitudinal) gauge metric $ds^2 = a_{\mathrm{FLRW}}^2(t) \left(-(1 + 2\Phi) dt^2 + (1 - 2\Psi) d\mathbf{x}^2 \right)$ with $\Phi = \Psi = -C/3$, $a_{\mathrm{FLRW}} = a_{\mathrm{EdS}}$.

Initial conditions and probability distribution

- Straightforward calculations
  $\Rightarrow$ dimensionless quantities at $t_{\rm in} = 0$ satisfy
  $\lim_{t\to 0} \frac{a}{t^{2/3}} = 1, \quad \hat{R}_{\rm in}(x) = -\frac{20}{9} S(x),$
  $(\hat{\sigma}_{\rm in})^i_j(x) = 0, \quad (\hat{r}_{\rm in})^i_j(x) = -\frac{5}{9}\delta^{ik} s_{kj}(x)$, where
- $\partial_i \partial_j C(x) = s_{ij} + \frac{1}{3}\delta_{ij} S$ is a symmetric Gaussian random matrix (invariant under orthogonal conjugation).
- Using the theory of such matrices one finds

$$p(S, \delta, \varphi) \sim e^{-\frac{1}{10}S^2 - \frac{1}{2}\delta^2} \delta^4 \,|\sin(3\varphi)|, \text{ where}$$

  - δ, φ parametrize the eigenvalues δ_k of s_{ij} via

$$\delta_k = \frac{2}{3}\delta \cos(\varphi + \frac{2\pi k}{3});$$

  - the normalization involves the value of $\langle(\nabla^2 C)^2\rangle$: an integral requiring an ultraviolet cutoff.

Volume evolution

$$V_{\mathcal{D}}(t; S_b, \Lambda) \sim \int a^3(t; S, \delta, \varphi, \Lambda) e^{-\frac{1}{10}(S-S_b)^2 - \frac{1}{2}\delta^2} \delta^4 \, |\sin(3\varphi)| dS \, d\delta \, d\varphi$$

can be computed numerically, using the evolution equations for $a(t; S, \delta, \varphi, \Lambda)$; implies

- volume scale factor $a_{\mathcal{D}}(t) = V_{\mathcal{D}}^{1/3}(t)$,
- Hubble rate $H_{\mathcal{D}}(t) = \dot{a}_{\mathcal{D}}(t)/a_{\mathcal{D}}(t)$,
- deceleration parameter $q_{\mathcal{D}}(t) = -\ddot{a}_{\mathcal{D}} a_{\mathcal{D}} / \dot{a}_{\mathcal{D}}^2$.

For $\Lambda = 0$ these do not agree with what we observe (even though $q_{\mathcal{D}} < 0$ is possible for positive background curvature).

A formula for structure distance

- Angular diameter distance d_A and luminosity distance d_L are related by $d_L/d_A = (1+z)^2$ (Etherington).
- Sachs equations $\Rightarrow$ the "structure distance" (Weinberg) $d_S := (1+z)d_A = (1+z)^{-1}d_L$ obeys

$$\ddot{d}_S + H_\sharp \dot{d}_S + i\, d_S = 0 \qquad \text{with}$$

  - $H_\sharp = -\frac{d}{dt}\ln(1+z)$,
  - $i = (1+z)^{-2}(|\sigma_{\text{opt}}|^2 + \frac{1}{2}R_{\alpha\beta}k^\alpha k^\beta) - \frac{d^2}{dt^2}\ln(1+z)$ (σ_{opt} ... shear of the null bundle).
- $d_{S\sharp} = \int_{t_e}^{t_o}(1+z)dt = \int_0^z \frac{1}{-[\ln(1+z)]^{\cdot}}dz$ is a solution for $i=0$ (holds e.g. for flat homogeneous universes!).

Inferred Hubble rate and deceleration

- "Hubble rates" $H_{\rm inf} = \frac{dz}{dd_S}$, $H_\sharp = \frac{dz}{dd_{S\sharp}} = -\frac{d}{dt}\ln(1+z)$,
  - $\frac{H_\sharp}{H_{\rm inf}} = 1 - \int_t^{t_o} \frac{i}{(1+z)} d_S\, dt'$,
  - $i > 0$ ($i < 0$): observations over-(under-)estimate expansion rates in previous epochs,
- inferred time parameter $dt_{\rm inf} = -\frac{dd_S}{1+z} = -\frac{\dot{d}_S}{1+z}dt$,
  - $H_{\rm inf}\, dt_{\rm inf} = H_\sharp\, dt = -d\ln(1+z)$,
- deceleration $q_{\rm inf} = \frac{d}{dt_{\rm inf}}\left(\frac{1}{H_{\rm inf}}\right) - 1, \quad q_\sharp = \frac{d}{dt}\left(\frac{1}{H_\sharp}\right) - 1$,
  - $q_{\rm inf} = q_\sharp + i\,\frac{d_S(1+z)}{\dot{d}_S\ \dot{z}}$,
  - $i < 0$ can simulate acceleration.

Homogeneous and irrotational dust universes

Assume

- Metric $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = -dt^2 + g_{ij}(t,x)dx^i dx^j$,
- energy-momentum tensor with $T_{ij} = g_{ij}T^k_k/3$ and $T_{0i} = 0$.

Then (using $1 + z = dt/ds$ and geodesic equations) one finds

$$
\begin{aligned}
H_\sharp &= -\frac{d}{dt}\ln(1+z) = -\frac{\theta}{3} + \sigma_{ij}\dot{x}^i\dot{x}^j, \\
i &= (1+z)^{-2}|\sigma_{\mathrm{opt}}|^2 + R/6 - \sigma^2 \\
&\quad +(-\sigma_{ij}\theta - 2\sigma_i^k\sigma_{kj} - r_{ij} + 2\sigma_{ij}\sigma_{kl}\dot{x}^k\dot{x}^l)\dot{x}^i\dot{x}^j \\
&\quad +\dot{x}^i\partial_i\theta/3 + \dot{x}^k(\partial_k\sigma_{ij})\dot{x}^i\dot{x}^j - 2\sigma_{ij}\Gamma^i_{kl}\dot{x}^k\dot{x}^l\dot{x}^j.
\end{aligned}
$$

Photon path average: basic ideas

Idea: replace $H_\sharp$ and i by suitably defined averages; $\langle X\rangle_{\rm pp}(t)$ should be the average of X over all

- spatial positions **x** occupied by photons,
- directions **v** of propagation.

Photon paths correspond to curves in **x**–space.

- Flat homogeneus case: straight lines,
- if the shapes were not altered by inhomogeneities, then $p(\dots)$ would give the distribution of the basic parameters with respect to the euclidean metric $dl^2 = \delta_{ij} dx^i dx^j$ along such a curve;
- approximation: same distribution even in the general case.

Photon path average: definition

- From euclidean length to physical time:
  $v^i = \frac{dx^i}{dl} = \dot{x}^i \, \frac{dt}{dl}$ ... tangent vector with $\delta_{ij} v^i v^j = 1$,
  g-norm $\sqrt{g_{ij} v^i v^j}$, use $g_{ij} \dot{x}^i \dot{x}^j = 1 \Rightarrow dt = \sqrt{g_{ij} v^i v^j}\, dl$.
- For every euclidean path segment dl
  - average over the parameters of the model ($\langle \ldots \rangle_{\mathrm{mw}}$),
  - average over all directions $\mathbf{v}$ ($\int_{S^2} \ldots d^2 v$),
  - weight by the time $dt = \sqrt{g_{ij} v^i v^j}\, dl$ spent in the segment

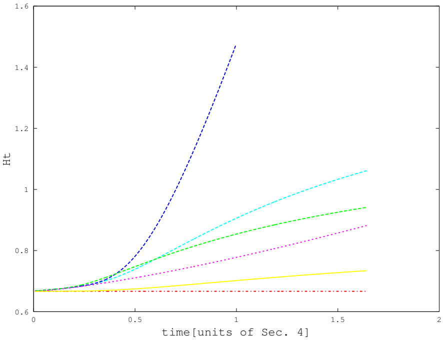
$$\Rightarrow \quad \langle X \rangle_{\mathrm{pp}} = \frac{\langle \int_{S^2} X \sqrt{g_{ij} v^i v^j} d^2 v \rangle_{\mathrm{mw}}}{\langle \int_{S^2} \sqrt{g_{ij} v^i v^j} d^2 v \rangle_{\mathrm{mw}}}.$$

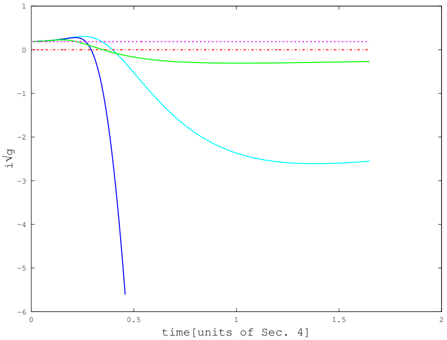
If X depends on $\dot{x}^i$ explicitly, use $\dot{x}^i = \frac{v^i}{\sqrt{g_{ij} v^i v^j}}$.

- Second order perturbation theory computations of $\langle H_\sharp \rangle_{\rm pp}$, $\langle i \rangle_{\rm pp}$ straightforward, but not all of the previous simplifications and approximations respect 2nd order PT.
- Non-perturbative results with GNU octave:
  - Euler method with logarithmic time steps,
  - constant $\hat{r} = \hat{r}_{\rm in}$,
  - a few further approximations;
- three scenarios for modelling collapse:
  1. collapse to half of maximal size, quantities retain values,
  2. collapse to half of maximal size, then ignore,
  3. ignore any region from the moment it starts collapsing

  (motivated by virial theorem, second one most realistic).
- Coding in plots: dashed lines for $H_\sharp$, $q_\sharp$, $d_{S\sharp}$, solid lines for other quantities from scenarios 1/2/3.

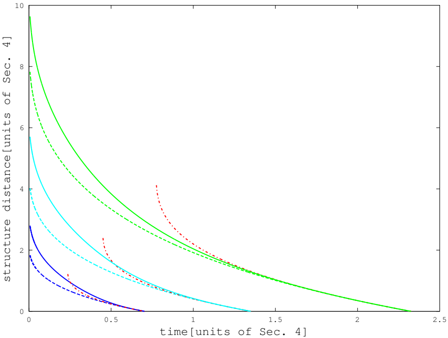
Time evolution of Ht and $i\sqrt{g}$

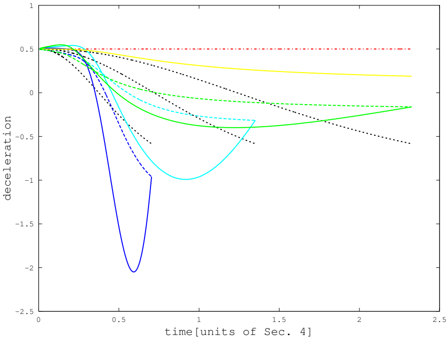




- Yellow line: result from volume averaging
- Strong deviations from homogeneous case: mainly from local anisotropy

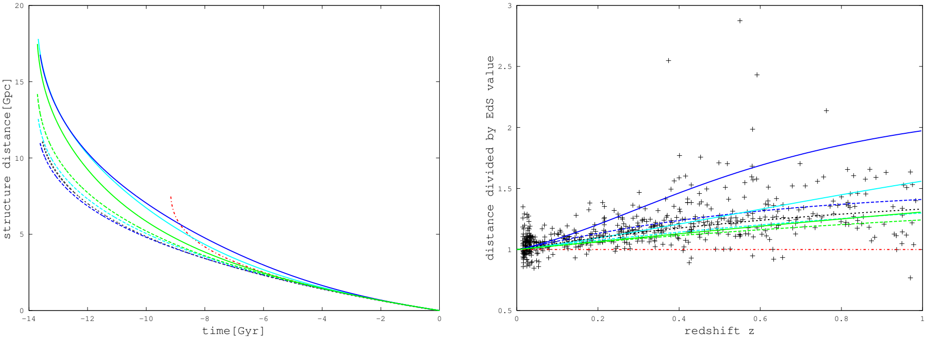
Time evolution of distance and deceleration





- t_o chosen such that $H_o t_o = 1$
- dotted black lines: ΛCDM

Distance in physical units



- Black crosses: observed supernovae
- Better agreement for smaller redshift

Last scattering

- A combination of our programs and linear perturbation theory gives $t_{\mathrm{ls}} \approx 5.3 \times 10^{-5}$ (the time at which $z = 1090$).
- Distance to CMB:
  - $d_S(t) = d_S(0) + d_S^{(1)} t^{1/3} + \mathcal{O}(t^{2/3})$ near $t = 0$
  - results in $d_S(t_{\mathrm{ls}}) \approx 20.7/20.9/19.8\ \mathrm{Gpc}$ for scenarios 1/2/3;
  - overestimation by 50% (compared to Planck data): σ_{opt}-term or measured vs computed d_A?
- Density perturbations (total):

$$\left(\frac{\Delta\rho}{\rho}\right)_{\mathrm{ls}} = \frac{1}{2} t_{\mathrm{ls}}^{\frac{2}{3}} \Delta S = \frac{1}{2} \times (5.3 \times 10^{-5})^{\frac{2}{3}} \times \sqrt{5} \approx 1.6 \times 10^{-3}$$

  - Baryonic ones should satisfy $\Delta\rho_B/\rho_B = 3\Delta T/T$ (T ... temperature),
  - fits well with $\Delta T/T \approx 10^{-5}$, $\Delta\rho/\rho \approx 50\Delta\rho_B/\rho_B$ (structure formation).

Summary of results

Considering a universe that

- is matter dominated and obeys the Einstein equations,
- in its early stages was very close to being spatially flat and homogeneous, with only Gaussian perturbations, and
- has vanishing cosmological constant, $\Lambda = 0$,

volume evolution is hardly affected by the inhomogeneities, but there is a time t_o when observations would suggest

- an inferred Hubble rate $H_{\rm inf}$ such that $H_{\rm inf} t_o \approx 1$,
- an inferred deceleration parameter of $q_{\rm inf} \approx -0.5$, and
- density perturbations at a redshift of 1090 that fit well with structure formation.

Approximations used

- Irrotational dust,
- statistical quantities $\rightarrow$ expectation values,
- distribution as if photon paths were straight lines,
- simplified evolution for r_{ij},
- approximations for $\sqrt{g_{ij} v^i v^j}$
- neglect of σ_{opt}, i.e. Weyl focusing,
- numerical errors from discretization.

The second and third approximation violate 2^{nd} order perturbation theory $\rightarrow$ serious discrepancy with literature.