UNIVERSITY OF CANTERBURY
Te Whare Wānanga o Waitaha

Inhomogeneous Cosmologies workshop, Torun, 2017

The Lagrangian perturbation theory of large-scale structure formation in relativistic cosmologies: The pressure effect

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[L1] T. Buchert and M. Ostermann, *Lagrangian theory of structure formation in relativistic cosmology I: Lagrangian framework and definition of a nonperturbative approximation.* Phys. Rev. **D 86**, 023520 (2012).

[L2] T. Buchert, C. Nayet, and A. Wiegand, *Lagrangian theory of structure formation in relativistic cosmology II: average properties of a generic evolution model*. Phys. Rev. **D 87**, 123503 (2013).

[L3] A. Alles, T. Buchert, F. Al Roumi and A. Wiegand, *Lagrangian theory of structure formation in relativistic cosmology III: gravitoelectric perturbation and solution schemes at any order*. Phys. Rev. **D 92**, 023512 (2015).

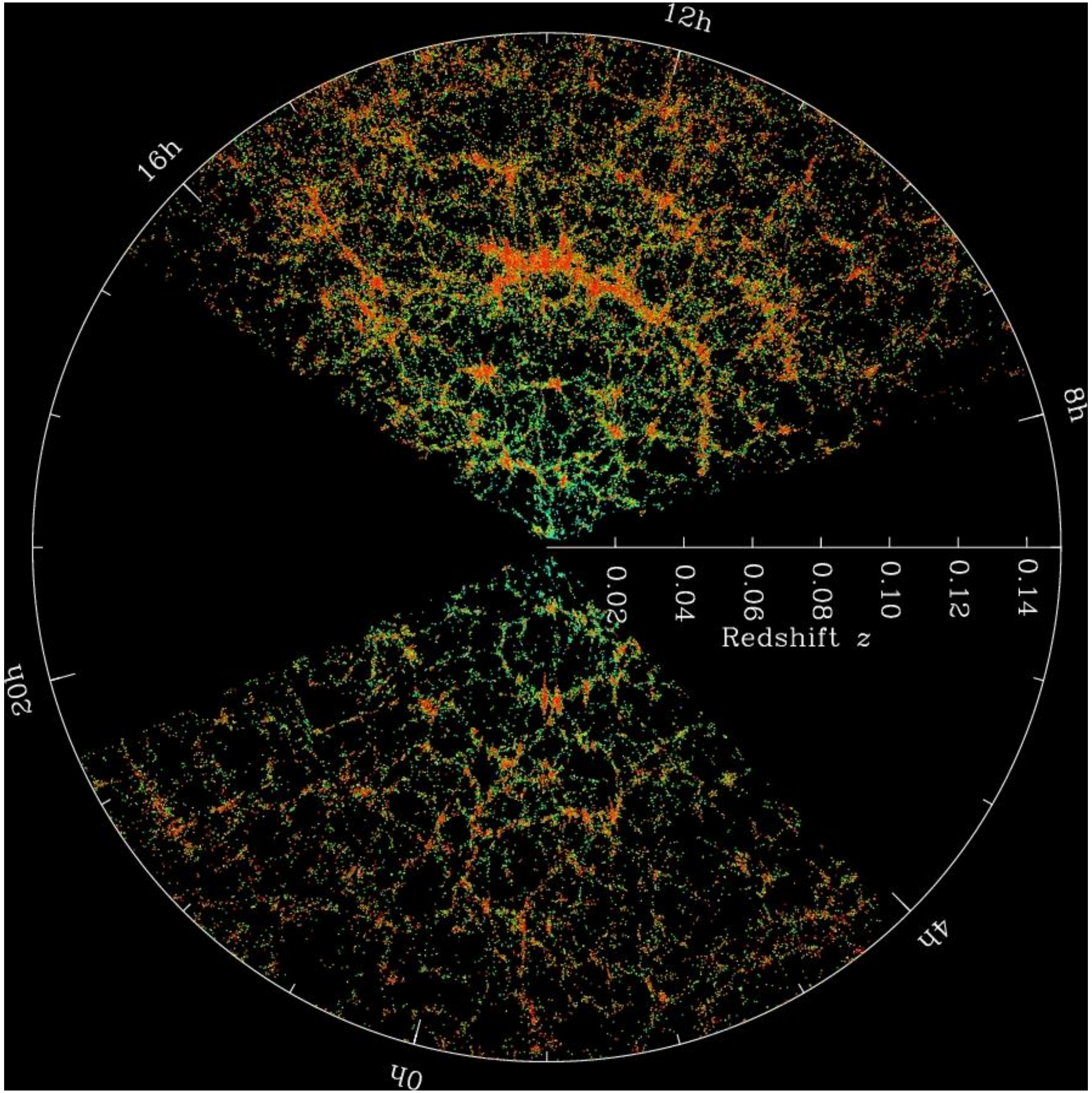
[L4] Thomas Buchert , Fosca Al Roumi, and Alexander Wiegand, *Lagrangian theory of structure formation in relativistic cosmology IV: Lagrangian approach to gravitational waves*, in preparing.

Large-scale structure

Understanding the large-scale structure of the universe is one of the main goals of the modern cosmology. Not only is it important for the evolution of Universe but also it can help us to uncover one of greatest mysteries of our Universe – Dark energy.

The large-scale distribution of galaxies observed by the Sloan Digital Sky Survey. Each dot is a galaxy; the colour bar shows the local density.

© SDSS collaboration



Newtonian cosmological perturbation theory (NCPT)

Newtonian perturbations with co-moving coordinates - simplified case

$$\frac{\partial \rho}{\partial t} + 3H\rho + \frac{1}{a}\nabla_x \cdot (\rho u) = 0,$$

$$\mathrm{r} = ax$$

$$\frac{\partial u}{\partial t} + Hu + \frac{1}{a}(u \cdot \nabla_x)u = -\frac{1}{a\rho}\nabla_x p - \frac{1}{a}\nabla_x \Phi,$$

$$\nabla_x^2 \Phi = 4\pi G \rho_H a^2 \delta\rho$$

The peculiar velocity is defined by

$$u = a\frac{dx}{dt} = v - Hr, \qquad v = \frac{dr}{dt}$$

The density contrast is defined by

$$\rho = \rho_H(1 + \delta\rho)$$

The evolution equation of density contrast

Difficult to solve analytically!!

$$\frac{\partial^2 \delta\rho}{\partial t^2} + 2H\frac{\partial \delta\rho}{\partial t} = \frac{1}{\rho a^2}\Delta p + \frac{1}{a^2}\nabla_x \cdot [(1 + \delta\rho)\nabla_x \Phi] + \frac{1}{a^2}\sum \partial_i \partial_j [(1 + \delta\rho)u_i u_j]$$

N-body simulations

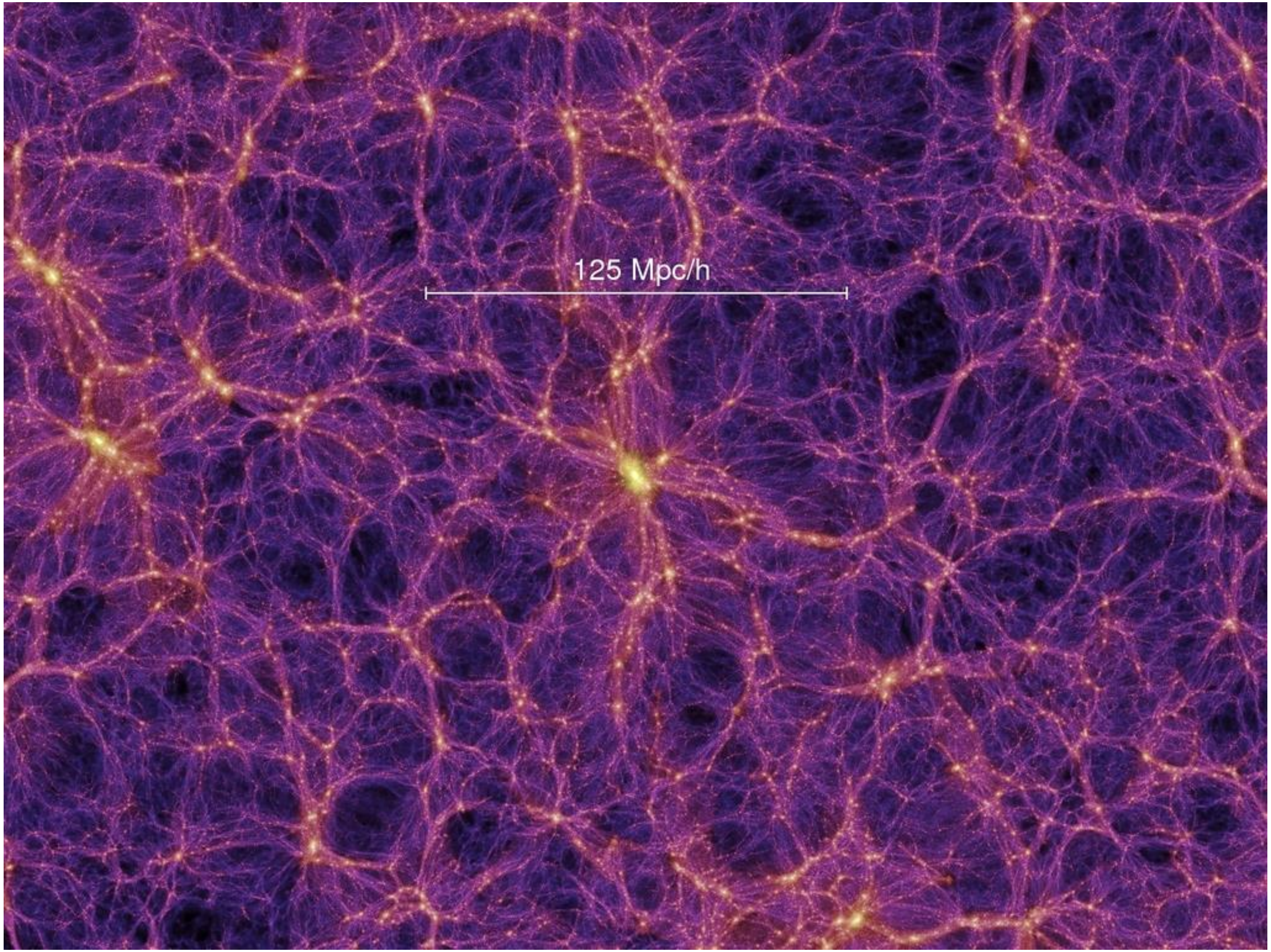
Z=0, t=13.6 Gyr

15 Mpc/h thick

Millennium Run

Max-Planck-Institut
für Astrophysik

Volker Springel et. al. 2004



Lagrangian perturbation theory (LPT)

Newtonian case

$$\frac{\partial \rho}{\partial t} + 3H\rho + \frac{1}{a}\nabla_x \cdot (\rho u) = 0, \qquad \frac{\partial u}{\partial t} + Hu + \frac{1}{a}(u \cdot \nabla_x)u = -\frac{1}{a\rho}\nabla_x p - \frac{1}{a}\nabla_x \Phi,$$

$$\Delta_x \Phi = 4\pi G \rho$$

Define the displacement field which describes the displacement of the fluid volume elements away from their initial positions.

$$\boldsymbol{x} = \boldsymbol{a}(\boldsymbol{t})(\boldsymbol{X} + \boldsymbol{P}(\boldsymbol{X}, t))$$

The transformation matrix

$$J_{ij}(X,t) := \delta_{ij} + \frac{\partial P_i(X,t)}{\partial X_j}$$

The mass conservation

$$\rho(\boldsymbol{X},t) = \left[\frac{a_0}{a}\right]^3 \rho(\boldsymbol{X},t_0) \frac{\det J_{ij}(\boldsymbol{X},t_0)}{\det J_{ij}(\boldsymbol{X},t)}$$

The mass density is analytically given by an equation of displacement field!!
The large-scale structure formation is described by the evolution equation of the displacement field!!

Euler-Jeans-Newton (EJN) model

where the velocity dispersion is approximated by an isotropic tensor field which is commonly modelled as a function of the local density

$$p = p(\rho)$$

The evolution equation of the displacement field is

$$\ddot{\boldsymbol{P}} + 2H\dot{\boldsymbol{P}} - 4\pi G\rho_H \boldsymbol{P} = \frac{\rho_H p'(\rho)}{a^2\rho}\Delta_{\mathbf{0}}\boldsymbol{P}$$

Δ_0: with respect to $\boldsymbol{X}$

Single-streamed flat dust model

$$\dot{\boldsymbol{P}} := \frac{\partial \boldsymbol{P}}{\partial t} \mid \boldsymbol{X} = \partial_t \boldsymbol{P} + \boldsymbol{a}^{-1}(\boldsymbol{U}\cdot\nabla)\boldsymbol{P}$$

The evolution equation of the displacement field is

$$\ddot{\boldsymbol{P}} + 2H\dot{\boldsymbol{P}} - 4\pi G\rho_H \boldsymbol{P} = \boldsymbol{0}$$

To solve evolution equations, suppose

$$\boldsymbol{P}(\boldsymbol{X},t) = \sum_{m=1}^{\infty} \varepsilon^{(m)}\boldsymbol{P}^{(m)}(\boldsymbol{X},t)$$

For dust case, the displacement field has the form

$$\boldsymbol{P}^{(m)}(\boldsymbol{X},t) \propto q^{(m)}(t)\boldsymbol{Q}^{(m)}(\boldsymbol{X})$$

For other cases, the spatial part of the displacement fields is related to the evolution equation so the form of the displacement fields is quite complicated.

linear solutions as $\boldsymbol{P} \propto D^+\,\alpha(\boldsymbol{X},0)$, $\boldsymbol{P} \propto D^-\,\beta(\boldsymbol{X},0)$

α, β are functions of position describing the initial data.

where $D^+ \propto t^{\frac{2}{3}}$, $D^- \propto t^{-1}$

see T. Buchert et. al's works.

Zel'dovich approximation

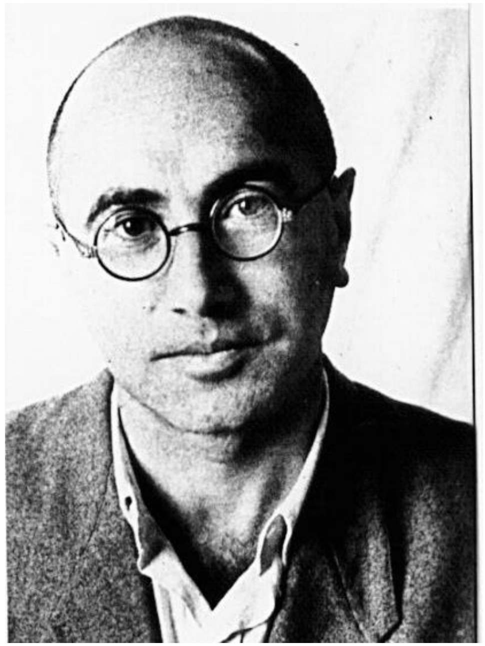
Applied the linear theory to the dust in the Newtonian approximation in terms of Lagrangian coordinates.

$$\boldsymbol{r} = a(t)\boldsymbol{q} + b(t)\boldsymbol{p}(\boldsymbol{q}).$$

By choosing the appropriate coordinates system, the conservation of mass is given by

$$\rho(a-\alpha b)(a-\beta b)(a-\gamma b) = \bar{\rho}a^3$$

α, β, γ are the three roots of $\left|\frac{\partial p_i}{\partial q_j} + \xi\delta_{ij}\right| = 0.$



Yakov Borisovich Zel'dovich
1914 - 1987

It is apparent that the Zel'dovich approximation is a special case of first order of Lagrangian perturbation theory (Buchert 1989).

- A&A 5, 84-89 (1970).
- Rev. Mod. Phys. 61, 2 (1989).

Even for not small mass density perturbations, the solutions of LPT is qualitatively correct . In other words, the LPT contains more non-linear information of large-scale structure than NCPT .

Summary: the differences between NCPT and LPT

- In a Lagrangian perturbation theory, only one variable, the trajectory field $\boldsymbol{x} = \boldsymbol{f}\ (\boldsymbol{X}, t)$, is perturbed.

- The first order Lagrangian approximations covers the so-called Zel'dovich approximation with some restrictions. (Buchert et.al. RZA1, 2, 3)

- A Lagrangian perturbation solution includes nonlinear terms thus contains more nonlinear information than Eulerian perturbation solution.

- A problem is the shell crossing, where different particles may have same physical positions. *Adhesion approximation* is developed trying to solve this problem (A&A 438 (2005) 443)

LPT in relativistic cosmologies – the perfect fluid

The two conservation laws in Newtonian case are replaced by three conservation laws:

the rest mass density conservation: $\partial_t \rho + \theta \rho = 0$

the energy density conservation: $\partial_t \epsilon + \theta(\epsilon + p) = 0$

the momentum conservation: $a_i + \frac{1}{\epsilon + p} h_i^j \, \partial_j p = 0$

energy-momentum tensor

$$T_{\alpha\beta} = (\epsilon + p) u_\alpha u_\beta + p h_{\alpha\beta}$$

expansion rate

$$\theta = \nabla^\alpha u_\alpha$$

acceleration field

$$a_\alpha = u^\beta \nabla_\beta u_\alpha$$

the metric is given as follow using the Lagrangian coordinates

$$ds^2 = -N^2 dt^2 + g_{ij} dX^i dX^j$$

$N = \frac{\rho}{\epsilon + p}$ is lapse function, and $g_{ij} = h_{ij}$ is spatial metric.

Dynamical equations

Constraint equations:

$$\mathcal{R} + K^2 - K^i_j \; K^j_i = 16\,\pi\,G\,\epsilon$$

$\mathcal{R}$ is the spatial Ricci scalar

$$D_i K^i_j - \partial_j K = 0$$

K is the trace of extrinsic curvature

"D" means the covariant derivative with respect to the spatial metric.

Evolution equations:

$$\frac{\partial}{\partial t} h_{\alpha\beta} = -2NK_{\alpha\beta},$$

$$K = -\theta$$

$$\frac{\partial}{\partial t} K^\alpha_\beta = N[\mathcal{R}^\alpha_\beta + KK^\alpha_\beta - 4\pi G \delta^\alpha_\beta(\epsilon - p) - (D_\beta a^\alpha + a^\alpha a_\beta)],$$

To develop LPT in relativistic cosmologies, one can use Cartan's co-frames $\boldsymbol{\eta}^a$, a non-coordinate basis of 3-dimensional space, so the spatial part of metric can be rewritten as

$$\boldsymbol{g} = \delta_{ab}\boldsymbol{\eta}^a \otimes \boldsymbol{\eta}^b = g_{ij} dX^i \otimes dX^j$$

and define the deformation one-form fields, $P^a_i(\boldsymbol{X}, t)$

$$\boldsymbol{\eta}^a = \eta^a_i dX^i, \qquad \eta^a_i = a(t)[\delta^a_i + P^a_i(\boldsymbol{X}, t)]$$

In this case, the field corresponding to the Newtonian Jacobian matrix:

$$J = \det(\eta^a_i) = \frac{1}{6}\varepsilon_{abc}\varepsilon^{ikl}\eta^a_i\eta^b_k\eta^c_l$$

the expansion rate and the mass density conservation are

$$\theta = \frac{1}{2J}\varepsilon_{abc}\varepsilon^{ikl}\dot{\eta}^a_i\eta^b_k\eta^c_l, \qquad \rho J = \rho_\circ J_\circ,$$

the dynamical equations of the perfect fluid can also be represented by co-frames.

Assumptions and initial conditions

assumptions:

- Friedmann equations still hold for background spacetime.

$$3(\frac{\partial_t a}{a})^2 = 8\pi G\epsilon_H, \quad 3\frac{\partial_t^2 a}{a} = -4\pi G(\epsilon_H + 3p_H), \quad b = \frac{\rho_H}{\epsilon_H + p_H}.$$

- Pressure is only the function of energy density, e. g. $p = p(\epsilon) = w\epsilon$.

initial conditions at time t_0:

- $N_\circ = b_\circ(1 + \delta N_\circ)$, $\rho_\circ = \rho_{H\circ}(1 + \delta\rho_\circ)$, so do $\epsilon_\circ$ and $p_\circ$.
- $P(\boldsymbol{X}, t_0) \neq 0$ so that at initial time $\hat{\eta}_i^a \neq \delta_i^a$.
- $a(t_0)$=1.

The evolution equation for the trace of displacement field for perfect fluid:

$$\partial_t^2 P + 2H(1-3w)\partial_t P - 4\pi G(\epsilon_H + 3p_H)(1-w)b^2 P = a^{-2} b^2 w \Delta_0 P$$

where $\Delta_0 = \delta^{ij}\partial_i\partial_j$ is the linearized trace of the Laplace-de Rham operator on the manifold.

However, this evolution equation is only valid for small deformation fields. To understand the physical meaning of this equation, much more work need to be done.

Special solutions:

$w = 0$: dust case

For this case, the evolution equation reduce to the well-known dust equation:

$$\partial_t^2 P + 2H\partial_t P - 4\pi G \epsilon_H P = 0$$

with the solution

$$\boldsymbol{P} = D^+ \, \alpha(\boldsymbol{X}, 0), \qquad \boldsymbol{P} = D^- \, \beta(\boldsymbol{X}, 0)$$

$$D^+ = t^{\frac{2}{3}}, \qquad D^- = t^{-1}$$

Special solutions: $w \neq 0$ perfect fluids case

Unlike in Newtonian cosmologies, there is no global coordinate system in relativistic universe which means Fourier transformation is not invertible.

Consider the solution has the form

$$P(t, \boldsymbol{X}) = P_K(t, K) Q^{(K)}(\boldsymbol{X}, \boldsymbol{K})$$

$$\Delta_0 P_K(t, K) = 0, \qquad \Delta_0 Q^{(K)}(\boldsymbol{X}, \boldsymbol{K}) = -K^2 Q^{(K)}(\boldsymbol{X}, \boldsymbol{K})$$

$$|\boldsymbol{K}|^2 = K^2$$

Helmholtz equation

The temporal part becomes

$$\partial_t^2 P_K + 2H(1-3w)\partial_t P_K - 4\pi G(\epsilon_H + 3p_H)(1-w)b^2 P_K = -a^{-2} b^2 w K^2 P_K$$

The solutions are

$$P_K = D_K^{(+)} C_1 + D_K^{(-)} C_2$$

$$D_K^{(\pm)} = t^{\alpha} J_{\pm\beta}\left(\frac{2\sqrt{a_3 w K^2}}{\nu} t^{\frac{\nu}{2}}\right)$$

$C_{1(2)}$
are two arbitrary functions of K and only depend on the initial condition.

$$\alpha = -\frac{1-9w}{6(1-w)}, \beta = \frac{3w+5}{2(1+3w)}, \nu = \frac{2(1+3w)}{3(1-w)}, a_3 = b_i^2 a_i^{-2} t_i^{\frac{4(1-3w)}{3(1-w)}}$$

Summaries

- Can be reduced to dust case.

- But: solutions can be obtained in this case by the strategy:
  - solving the Eulerian problem in global coordinates using Fourier transformation
  - performing inverse Fourier transformation to obtain an Eulerian solution in terms of x
  - replacing formally x by X, the so-constructed solution solves the Riemannian problem by construction, since the equation is algebraically identical.

- ....

Anisotropic stress?

One can consider anisotropic pressure

$$T_{\mu\nu} = (\epsilon + p)u_\mu u_\nu + p g_{\mu\nu} + \pi_{\mu\nu}$$

It can be easily imaged that with the appearance of anisotropic pressure, the situation becomes even more complicated. In this case the fluids are not perfect fluids and their equations of state are not clear until we choose some special ones. A hard task in this case is to find the way to relate the physical dynamics to the deformation fields.

Thank you for your attention!